Strong connectivity and directed triangles in oriented graphs. Partial results on a particular case of the Caccetta-Häggkvist conjecture

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Abstract

A particular case of Caccetta-Häggkvist conjecture, says that a digraph of order n with minimum out-degree at least $\frac{1}{3}n$ contains a directed cycle of length at most 3. In a recent paper, Kral, Hladky and Norine (see [7]) proved that a digraph of order n with minimum out-degree at least $0.3465n$ contains a directed cycle of length at most 3 (which currently is the best result). A weaker particular case says that a digraph of order n with minimum semi-degree at least $\frac{1}{3}n$ contains a directed triangle. In a recent paper (see [8]), by using the result of [7], the author proved that for $\beta \geq 0.343545$, any digraph D of order n with minimum semi-degree at least βn contains a directed cycle of length at most 3 (which currently is the best result). This means that for a given integer $d \geq 1$, every digraph with minimum semi-degree d and of order md with $m \leq 2.91082$, contains a directed cycle of length at most 3. In particular, every oriented graph with minimum semi-degree d and of order md with $m \leq 2.91082$, contains a directed triangle. In this paper, by using again the result of [7], we prove that every oriented

graph with minimum semi-degree d, of order md with $2.91082 < m \leq 3$ and of strong connectivity at most $0.679d$, contains a directed triangle. This will be implied by a more general and more precise result, valid not only for $2.91082 < m \leq 3$ but also for larger values of m. As application, we improve two existing results. The first result (Authors Broersma and Li in [2]), concerns the number of the directed cycles of length 4 of a triangle free oriented graph of order n and of minimum semi-degree at least $\frac{n}{3}$. The second result (Authors Kelly, Kühn and Osthus in [10]), concerns the diameter of a triangle free oriented graph of order n and of minimum semi-degree at least $\frac{n}{5}$

Keywords : Oriented graph, strong connectivity, girth, triangle

1 Introduction and definitions

The definitions which follow are those of [1].

We consider digraphs without loops and without parallel arcs. $V(D)$ is the *vertex set* of D and the *order* of D is the cardinality of $V(D)$. $\mathcal{A}(D)$ is the set of the arcs of D. We denote by $a(D)$ the number of the arcs of D (*size* of D). Two arcs (x, y) and x', y' are *independent* if the pairs $\{x, y\}$ and $\{x', y'\}$ are disjoint.

We say that a vertex y is an *out-neighbor* of a vertex x (*in-neighbour* of x) if (x, y) (resp. (y, x)) is an arc of D. $N^+_D(x)$ is the set of the out-neighbors of x and $N^-_D(x)$ is the set of the in-neighbors of x. The cardinality of $N^+_D(x)$ is the *out-degree* $d^+_D(x)$ of x and the cardinality of $N^-_D(x)$ is the it in-degree $d^-_D(x)$ of x. We also put $N_D(x) = N^+_D(x) \cup N^-_D(x)$ and $N'_D(x) = N^+_D(x) \cup N^-_D(x) \cup \{x\}$. When no confusion is possible, we omit the subscript D. We denote by $\delta^+(D)$ the minimum out-degree of D and by $\delta^-(D)$ the minimum in-degree of D. The *minimum semi-degree* of D is $\delta^0(D) = \min\{\delta^+(D), \delta^-(D\}$.

For a vertex x of D and for a subset S of $V(D)$, $N^+_S(x)$ is the set of the out-neighbors of x which are in S, and $d^+_S(x)$ is the cardinality of $N^+_S(x)$. Similarly, $N^-_S(x))$ is the set of the

in-neighbors of x which are in S, and $d_S^-(x)$ is the cardinality of $N_S^-(x)$.

A *directed path* of length p of D is a list $x_0, \ldots, x_p$ of distinct vertices such that $(x_{i-1}, x_i) \in \mathcal{A}(D)$ for $1 \leq i \leq p$. A *directed cycle* of length $p \geq 2$ is a list $(x_0 \ldots, x_{p-1}, x_0)$ of vertices with $x_0 \ldots, x_{p-1}$ distinct, $(x_{i-1}, x_i) \in \mathcal{A}(D)$ for $1 \leq i \leq p-1$ and $(x_{p-1}, x_0) \in \mathcal{A}(D)$. From now on, we omit the adjective " directed". A p-cycle of D is a directed cycle of length p. A *digon* is a 2-cycle, and a triangle is a 3-cycle of D of length 3. The *girth* $g(D)$ of D is the minimum length of the cycles of D. The digraph D is said to be *strongly connected* (for briefly strong) if for every distinct vertices x and y of D, there exists a path from x to y. It is known that in a non-strong digraph D, there exists a partition (A, B) of $V(D)$ with $A \neq \emptyset$ and $B \neq \emptyset$ such that there are no arcs from a vertex of B to a vertex of A. (one say that A dominates B). We say that a subset S of $V(D)$ disconnects D, if the digraph $D - S$ is non-strong. The *strong connectivity* $k(D)$ of D is the smallest of the positive integers m such that there exists a subset of $V(D)$ of cardinality m disconnecting D. D is said to be *p-strong connected* if $k(D) \geq p$. It is well known that in a p-strong connected digraph, if S is a subset of $V(D)$ such that $|S| \geq p$ and $|V(D) \setminus S| \geq p$, then there exist p independent arcs with starting vertices in S and with ending vertices in $V(D) \setminus S$.

In a strong digraph D, for vertices x and y of D, the *distance* $d(x, y)$ from x to y is the length of a shortest path from x to y. The *diameter* $\operatorname{diam}(D)$ is the maximum of the distances $d(x, y)$. The *eccentricity* $\operatorname{ecc}(x)$ of a vertex x is the maximum of the distances $d(x, y)$, $y \in V(D)$. It is clear that $\operatorname{ecc}(x) \leq \operatorname{diam}(D)$ for every vertex x of D.

An *oriented graph*, is a digraph D such that for any two distinct vertices x and y of D, at most one of the ordered pairs (x, y) and (y, x) is an arc of D. The author proved in [9] that the strong connectivity k of an oriented graph D of order n, satisfy $k \geq \dfrac{2(\delta^+(D) + \delta^-(D) + 1) - n}{3}$, and this shows that an oriented graph of order n and of

minimum semi-degree at least $\frac{n}{4}$, is strongly connected.

Caccetta and Häggkvist (see [3]) conjectured in 1978 that the girth of any digraph of order n and of minimum out-degree at least d is at most $\lceil n/d \rceil$.

The conjecture is still open when $d \geq n/3$, in other words it is not known if any digraph of order n and minimum out-degree at least $n/3$ contains a cycle of length at most 3.

In fact it is also unknown if any digraph of order n with both minimum out-degree and minimum in-degree at least $n/3$ contains a cycle of length at most 3 and then a special case of the Caccetta-Häggkvist conjecture is :

Conjecture 1.1 *Every digraph of order* n *and of minimum semi-degree at least* $\frac{n}{3}$, *contains a cycle of length at most* 3.

Two questions were naturally raised :

Question $\mathbf{Q}_1$ What is the minimum constant c such that any digraph of order n with minimum out-degree at least cn contains a cycle of length at most 3.

Question $\mathbf{Q}_2$ What is the minimum constant c' such that any digraph of order n with both minimum out-degree and minimum in-degree at least $c'n$ contains a cycle of length at most 3.

It is known that $c \geq c' \geq 1/3$ and the conjecture is that $c = c' = 1/3$. In a very recent paper (See [7]), Hladký, Král' and Norine proved that $c \leq 0.3465$, which currently is the best result.

By using this result, the author proved in [8] that $c' \leq 0.343545$, which currently is the best result. In other terms, this means :

Theorem 1.2 *For* $d \geq 1$, *any digraph with minimum semi-degree* d *and of order at most* $2.91082d$ *contains a cycle of length at most* 3.

In our paper, we will see that in an oriented graph D of minimum semi-degree d and of order md with $2.91082 < m < \frac{2}{c}$, an adequate upper bound on the connectivity of D forces the existence of a triangle. More precisely, we prove :

Theorem 1.3 *Let D be an oriented graph of minimum semi-degree d, of order $n = md$ with $2.91082 < m < \frac{2}{c}$. If the connectivity k of D verifies $k \leq \max\left\{\frac{5-m-4c+c^2}{(1-c)(2-c)}d, \frac{2-cm}{2-c}d\right\}$, then D contains at least a triangle.*

Since $c \leq 0.3465$, an easy consequence will be :

Theorem 1.4 *Let D be an oriented graph of minimum semi-degree d, of order $n = md$ with $2.91082 < m \leq 3$. If the connectivity k of D verifies $k \leq 0.679d$, then D contains at least a triangle.*

Broersma and Li proved in [2] that in a triangle-free oriented graph of order n and of minimum semi-degree at least $\frac{n}{3}$, every vertex is in more than $1 + \frac{n}{15}(11 - 4\sqrt{6})$ 4-cycles. We improve this result by proving :

Theorem 1.5 *Let D be a triangle-free oriented graph of minimum semi-degree d, of order $n = md$ with $m \leq 3$. Then every vertex x of D is contained in more than $\frac{2(5-m-4c+c^2)d}{(1-c)(2-c)} + (2-m)d + 1$ cycles such that two of these cycles have only the vertex x in common.*

If we allow distinct 4-cycles with others vertices than x in common, we give an even more spectacular improvement, by proving :

Theorem 1.6 *Let D be a triangle-free oriented graph of minimum semi-degree d, of order $n = md$ with $m \leq 3$.*
Then every vertex x of D is contained in more than $\frac{11 - 15c + 7c^2 - c^3 - (c^2 - 3c + 3)m}{(1-c)^2(2-c)}d$ 4-cycles.

Kelly, Kühn and Osthus proved in [10] that if D is an oriented graph of order n and of minimum semi-degree greater than $\frac{n}{5}$, then either the diameter of D is at most 50 or D contains a triangle. We will considerably improve this result by proving :

Theorem 1.7 *If D is a triangle-free oriented graph of minimum semi-degree d and of order $n = md$ with $m \leq 5$, then the diameter of D is at most 9.*

A result of Chudnovsky, Seymour and Sullivan (see[5]) asserts that one can delete k edges from a triangle-free digraph D with at most k non-edges to make it acyclic. Hamburger, Haxell, and Kostochka used this to prove in [6] that in a triangle-free digraph D with at most k non-edges, $\delta^+(D) < \sqrt{2k}$ (and $\delta^-(D) < \sqrt{2k}$ also) .
Chen, Karson, and Shen improved in [4] the initial result of [5] by asserting that one can delete $0.8616k$ edges from a triangle-free digraph D with at most k non-edges to make it acyclic. From this result, by using the reasoning of Hamburger, Haxell and Kostochka in [6], it is easy to prove that in a triangle-free digraph D with at most k non-edges, $\delta^+(D) < \sqrt{1.7232k}$ and $\delta^-(D) < \sqrt{1.7232k}$. As the maximum size of an oriented graph of order n is $\frac{n(n-1)}{2}$, an immediate consequence is :

Lemma 1.8 *If D is a triangle-free oriented graph of order n, then $a(D) < \dfrac{n^2}{2} - \dfrac{(\delta^+(D))^2}{1.7232}$ and $a(D) < \dfrac{n^2}{2} - \dfrac{(\delta^-(D))^2}{1.7232}$.*

2 Proofs of Theorems 1.3 and 1.4

By hypothesis, D is an oriented graph of minimum semi-degree d, of order $n = md$ with $2.91082 < m < \frac{2}{c}$ and of strong connectivity k We put $k' = \frac{k}{d}$. Let K be a set of k vertices disconnecting D. Then there exists a partition of $V(D) \setminus K$ into two subsets A and B, such that there are no arcs from a vertex of B to a vertex of A. Without loss of generality, we

may suppose that $|B| \leq |A|$. We put $a = \dfrac{|A|}{d}$ and $b = \dfrac{|B|}{d}$. Since $b \leq a$, it holds $b \leq \dfrac{m-k'}{2}$. First we claim that :

Lemma 2.1 *If D is triangle-free, then for every arc (y,x) of D with $y \in A$ and $x \in B$, it holds $d^+_B(x) + d^-_A(y) \geq 2d - k'd$.*

Proof. Since x has no out-neighbors in A, x has $d^+(x) - d^+_B(x)$ out-neighbors in K, which means $|N^+_K(x)| = d^+(x) - d^+_B(x)$. Since y has no in-neighbors in B, y has $d^-(y) - d^-_A(y)$ in-neighbors in K, which means $|N^-_K(y)| = d^-(y) - d^-_A(y)$. Since $N^+_K(x)$ and $N^-_K(y)$ are vertex-disjoint (for otherwise, we would have a triangle), we have $d^+(x) - d^+_B(x) + d^-(y) - d^-_A(y) \leq k'd$, hence $d^+_B(x) + d^-_A(y) \geq d^+(x) + d^-(y) - k'd$ and since $d^+(x) \geq d$ and $d^-(y) \geq d$, the result follows ■

Now, we claim :

Lemma 2.2 *Suppose that $2.91082 < m < 5 - 4c + c^2$. If the connectivity k of D verifies $k \leq \dfrac{5 - m - 4c + c^2}{(1-c)(2-c)}d$, then D contains at least a triangle.*

Proof. We put $k' = \frac{k}{d}$. Suppose, for the sake of a contradiction, that D does not contain triangles. Let sd be the minimum out-degree of $D[B]$, and let x be a vertex of B with $d^+_B(x) = sd$. It is easy to verify that $\dfrac{5 - m - 4c + c^2}{(1-c)(2-c)} < 1$ and since all the out-neighbors of x are in $B \cup K$, it follows that $N^+_B(x) \neq \emptyset$, and so $s > 0$. There exists a vertex x' of $N^+_B(x)$, such that $d^+_{N^+_B(x)}(x') < csd$. It follows that x' has more than $(s - cs)d = (1-c)sd$ out-neighbors in B but not in $N^+_B(x)$, and these out-neighbors cannot be in-neighbors of x (for otherwise, we would have a triangle). We get then $d^-_{B \cup K}(x) < [b + k' - 1 - (1-c)s]d$. Suppose that $b + k' - 1 \geq 1$. Then $k' \geq 2 - b$, and since $b \leq \dfrac{m-k'}{2}$, we get $k' \geq 2 - \dfrac{m-k'}{2}$, hence $k' \geq 4 - m$. Then, since $k' \leq \dfrac{5 - m - 4c + c^2}{(1-c)(2-c)}$, we get $4 - m \leq \dfrac{5 - m - 4c + c^2}{(1-c)(2-c)}$, hence $(4-m)(c^2 - 3c + 2) \leq 5 - m - 4c + c^2$. This yields $m(c^2 - 3c + 1) \geq 3c^2 - 8c + 3$, hence $m(c^2 - 3c + 1) \geq 3(c^2 - 3c + 1) + c$. Since $c^2 - 3c + 1 > 0$, we get $m \geq 3 + \dfrac{c}{c^2 - 3c + 1}$. It

is easy to verify that for $\frac{1}{3} \leq c \leq 0.3465$, it holds $\dfrac{c}{c^2 - 3c + 1} > 1$. We get then $m > 4$, and it is easy to verify that this is contradictory with $m < 5 - 4c + c^2$. Consequently, we have $b + k' - 1 < 1$. We deduce $d^-_{B \cup K}(x) < d$, which means that $N^-_A(x) \neq \emptyset$ (in fact, by the above reasoning, this is true for every vertex of B). More precisely, we have

$$d^-_A(x) > [2 - k' - b + (1 - c)s]d \qquad (1)$$

There exists a vertex y of $N^-_A(x)$ with fewer than $cd^-_A(x)$ in-neighbors in $N^-_A(x)$ (for otherwise $D[N^-_A(x)]$ would contain a triangle). It follows $d^-_A(y) < cd^-_A(x) + ad - d^-_A(x)$, hence $d^-_A(y) < ad - (1 - c)d^-_A(x)$. From Lemma 2.1, we get $d^-_A(y) \geq (2 - k')d - d^+_B(x)$, that is $d^-_A(y) \geq (2 - k' - s)d$. We deduce $(2 - k' - s)d < ad - (1 - c)d^-_A(x)$, hence

$$sd > (2 - k' - a)d + (1 - c)d^-_A(x) \qquad (2)$$

From (1) and (2), we deduce $sd > (2 - k' - a)d + (1 - c)[2 - k' - b + (1 - c)s]d$, hence $s > 2-k'-a+2-2c-k'+ck'-b+bc+(1-c)^2 s$. It follows $(2c-c^2)s > 4-2k'-a-b-2c+ck'+bc$, and since $a + b = m - k'$, we get $(2c - c^2)s > 4 - m - k' - 2c + ck' + bc$. Since $s < bc$ (for otherwise $D[B]$ would contain a triangle), we get $(2c - c^2)bc > 4 - m - k' - 2c + ck' + bc$, hence $(1 - c)^2 bc < m + 2c - 4 + (1 - c)k'$. Since all the out-neighbors of x are in $B \cup K$, we have $1 - s \leq k'$, hence $s \geq 1 - k'$, and since $s < bc$, we get $bc > 1 - k'$. It follows $(1-k')(1-c)^2 < m+2c-4+(1-c)k'$, hence $k'(1-c)(2-c) > 1-2c+c^2-m-2c+4$. This implies $k' > \dfrac{5 - m - 4c + c^2}{(1 - c)(2 - c)}$, which is contradictory with the hypothesis on k. Consequently D contains at least a triangle, and so, the result is proved. ■

We claim also :

Lemma 2.3 *Suppose that* $2.91082 < m < \frac{2}{c}$. *If the connectivity* k *of* D *verifies* $k \leq \dfrac{2 - cm}{2 - c}d$, *then* D *contains at least a triangle.*

Proof. Suppose, for the sake of a contradiction, that D does not contain triangles. Let sd be the minimum out-degree of $D[B]$, and let x be a vertex of B with $d^+_B(x) = sd$. We have

then $k' \geq 1-s$, hence $s \geq 1-k'$. Since $s < bc$ (for otherwise we would have a triangle), we get $bc > 1-k'$. Since $b \leq \dfrac{m-k'}{2}$, it follows $\dfrac{(m-k')c}{2} > 1-k'$, hence $mc - k'c > 2-2k'$. It follows $k' > \dfrac{2-cm}{2-c}$, which is contradictory with the hypothesis on $k = k'd$. So, the result is proved. ■

It is easy to prove that $5-4c+c^2 < \frac{2}{c}$. By using these two lemmas, we get Theorem 1.3.

It is easy to see that we have $\dfrac{5-m-4c+c^2}{(1-c)(2-c)} \geq \dfrac{2-cm}{2-c}$ if and only if $m \leq \dfrac{3-2c+c^2}{1-c+c^2}$. Then Theorem 1.3 means that when $2.91082 < m \leq \dfrac{3-2c+c^2}{1-c+c^2}$, a strong connectivity not greater than $\dfrac{5-m-4c+c^2}{(1-c)(2-c)}d$ forces a triangle in D, and when $\dfrac{3-2c+c^2}{1-c+c^2} < m < \frac{2}{c}$, a strong connectivity not greater than $\dfrac{2-cm}{2-c}d$ forces a triangle in D.

It is easy to see that for $2.91082 < m \leq 3$, we have $m < \dfrac{3-2c+c^2}{1-c+c^2}$. Since $c \leq 0.3465$, it is easy to see that we have $0.679d < \dfrac{5-m-4c+c^2}{(1-c)(2-c)}d$. Then by Lemma 2.2, a strong connectivity no greater than $0.679d$ forces a triangle, and so Theorem 1.4 is proved. Since a digraph which is not oriented contains a digon, it is easy to see that proving Conjecture 1.1, amounts to proving that every oriented graph, of minimum semi-degree at least d, of order md with $2.91082 < m \leq 3$ and of connectivity $k > 0.679d$, contains at least a triangle.

3 Proofs of Theorems 1.5, 1.6 and 1.7

a) Proof of Theorem 1.5

By hypothesis D is a triangle-free oriented graph of minimum semi-degree d, of order $n = md$ with $m \leq 3$, and x is a vertex of D. Let k be the strong connectivity of D (and $k' = \frac{k}{d}$). We have $k > 0$ (for otherwise, by Theorem 1.3 we would have triangles). Clearly, we have $d^+(x) + d^-(x) < md$, and since $k \leq d^-(x)$, it follows $d^+(x) + k < md$, hence $md - d^+(x) > k$. As we have also $d^+(x) \geq k$, there exist k independent arcs $(y_1, z_1), \ldots, (y_k, z_k)$ with

$y_i \in N^+(x)$, $z_i \notin N^+(x)$ and $z_i \neq x$ for $1 \leq i \leq k$. Since D is triangle-free, we have also $z_i \notin N^-(x)$ for $1 \leq i \leq k$. It follows that the set $S_1 = \{z_1, \ldots, z_k\}$ is contained in $V(D)\backslash N'(x)$. Similarly, there exist k independent arcs $(v_1, u_1), \ldots, (v_k, u_k)$ with $u_i \in N^-(x)$, $v_i \notin N^-(x)$ and $v_i \neq x$ for $1 \leq i \leq k$. Since D is triangle-free, we have also $v_i \notin N^+(x)$ for $1 \leq i \leq k$. It follows that the set $S_2 = \{v_1, \ldots, v_k\}$ is contained in $V(D) \setminus N'(x)$. We have $|S_1 \cap S_2| = |S_1| + |S_2| - |S_1 \cup S_2|$. Since $|S_1| = |S_2| = k'd$ and $|S_1 \cup S_2|$ is contained in $V(D) \setminus N'(x)$, it follows $|S_1 \cap S_2| \geq 2k'd - (md - d^+(x) - d^-(x) - 1)$, hence $|S_1 \cap S_2| \geq 2k'd - md + d^+(x) + d^-(x) + 1$. Since $d^+(x) \geq d$ and $d^-(x) \geq d$, it follows $|S_1 \cap S_2| \geq (2k' + 2 - m)d + 1$. This implies the existence of at least $(2k' + 2 - m)d + 1$ 4-cycles containing x and such that any two of these cycles have only x in common. Now since D is triangle-free, we deduce from Theorem 1.3 that $k' > \dfrac{5 - m - 4c + c^2}{(1-c)(2-c)}$, and then Theorem 1.5 is proved.

Since $c \leq 0.3465$ and $m \leq 3$, it is easy to see that the number $n_D(x, 4)$ of 4-cycles of D containing x, and such that any two of these cycles have only x in common, is at least $\dfrac{2 \times (5 - 3 - 4 \times 0.3465 + 0.3465^2)d}{0.6535 \times 1.6535} - d + 1$, hence $n_D(x, 4) > 0.358d + 1$, and since $d \geq \frac{n}{3}$ (n being the order of D), we get $n_D(x, 4) > 0.119n + 1$. Since $1 + \frac{n}{15}(11 - 4\sqrt{6}) \approx 1 + 0.08014n$ (exceeding value), it is clear that our result improve that of Broersma and Li.

b) Proof of Theorem 1.6

Let $k = k'd$ be the strong connectivity of D. By Theorem 1.4, we have $k > 0.679d$. Clearly the eccentricity $\mathrm{ecc}(x)$ of x is at least 3 (for otherwise,we would have a triangle). The author proved in [9] that the diameter of an oriented graph of order n and of minimum semi-degree at least $\frac{n}{3}$ is at most 4. By this result, we have $\mathrm{ecc}(x) \leq 4$, and consequently $3 \leq \mathrm{ecc}(x) \leq 4$. For $1 \leq i \leq \mathrm{ecc}(x)$ let R_i be the set of the vertices z of D such that $d(x, z) = i$. Since D is triangle-free, all the in-neighbors of x are in $R_3 \cup \cdots \cup R_{\mathrm{ecc}(x)}$.

We claim that $d^-_{R_3}(x) > d - \dfrac{m-2-k'}{1-c}d$ (Assertion (Ass)).

We observe first that $m-2-k' > 0$. Indeed, for an arbitrary vertex u of D, there exists $k'd$ independent arcs with starting vertices in $N^+(u)$ and ending vertices in $V(D) \setminus N^+(u)$. Since D is triangle-free these ending vertices are not in $N^-(u)$. It follows $2d + k'd < md$, hence $m-2-k' > 0$.

Suppose first that $\mathrm{ecc}(x) = 3$. Then all the in-neighbors of x are in R_3. This implies $d^-_{R_3}(x) \geq d$, and since $d > d - \dfrac{m-2-k'}{1-c}d$, the assertion (Ass) is proved.
Suppose now that $\mathrm{ecc}(x) = 4$. Since R_2 disconnects D, we have $r_2 \geq k'd$. Suppose first that $r_3 \geq d$. We have $r_4 = md - r_1 - r_2 - r_3 - 1$, hence $r_4 < md - d - k'd - d$, that is $r_4 < (m-2-k')d$. It follows $d^-_{R_3}(x) > d - (m-2-k')d$, and since $d - (m-2-k')d > d - \dfrac{m-2-k'}{1-c}d$, the Assertion (Ass) is proved. Suppose now that $r_3 < d$. Clearly, all the in-neighbors of a vertex of R_4 are in $R_3 \cup R_4$. It follows that every vertex of R_4 has at least $d - r_3$ in-neighbors in R_4. Since $D[R_3]$ is triangle-free, it holds $d - r_3 < cr_4$, hence $r_4 > \dfrac{d - r_3}{c}$, hence $r_4 > \dfrac{d - (md - r_1 - r_2 - r_4)}{c}$, which gives $r_4 > \dfrac{(1-m)d + r_1 + r_2 + r_4}{c}$. Since $r_1 \geq d$ and $r_2 \geq k'd$, we get $r_4 > \dfrac{(2-m+k')d + r_4}{c}$, hence $(1-c)r_4 < (m-2-k')d$, and then $r_4 < \dfrac{m-2-k'}{1-c}d$. It follows $d^-_{R_3}(x) > d - \dfrac{m-2-k'}{1-c}d$, which is the assertion (Ass). It is easy to see that an in-neighbor z of x which is in R_3 has an in-neighbor z_2 in R_2 and that z_2 has an in-neighbor z_1 in R_1. Then $C_z = (x, z_1, z_2, z, x)$ is a 4-cycle of D, containing x . It is clear that the cycles C_z, $z \in N^-_{R_3}(x)$ are distinct. Consequently the vertex x is contained in more than $d - \dfrac{m-2-k'}{1-c}d$ 4-cycles. Since $k > \dfrac{5-m-4c+c^2}{(1-c)(2-c)}d$ (By Theorem 1.3), the result follows.

Since $c \leq 0.3465$, $m \leq 3$ and $k' > 0.679$, it holds $d^-_{R_3}(x) > d - \dfrac{3-2-0.679}{1-0.3465}d$, hence $d^-_{R_3}(x) > 0.5087d$, hence $d^-_{R_3}(x) > 0.169n$. So D possess more than $0.169n$ 4-cycles containing x, which is much better that the result of Broersma and Li.

c) Proof of Theorem 1.7

By hypothesis D is a triangle-free oriented graph of minimum semi-degree d, of order $n = md$ with $m \leq 5$. Suppose, for the sake of a contradiction, that the diameter of D is at least 10. Then let x and y be two vertices of D such that $d(x, y) \geq 10$. For $1 \leq i \leq 6$, let R_i be the set of the vertices z of D such that $d(x, z) = i$, and for $1 \leq i \leq 3$, let R_{-i} be the set of the vertices z of D such that $d(z, y) = i$. For $1 \leq i \leq 6$, r_i is the cardinality of R_i and for $1 \leq i \leq 3$, r_{-i} is the cardinality of R_{-i}. The sets R_i, $1 \leq i \leq 6$ are mutually vertex-disjoint, the sets R_{-i}, $1 \leq i \leq 3$ are also mutually vertex-disjoint, and a set R_i, $1 \leq i \leq 6$ is a vertex-disjoint with a set R_{-j}, $1 \leq j \leq 3$ (for otherwise the diameter of D would be at most 9). For $2 \leq i \leq 6$ we put $R_i^{'} = R_1 \cup \cdots R_i$, for $2 \leq i \leq 3$ we put $R_{-i}^{'} = R_{-1} \cup \cdots R_{-i}$, and $r_i^{'}$, $r_{-i}^{'}$ are the respective cardinalities.

We claim that $r_3' \geq 2.239d$. Indeed, since $D[R_1]$ is triangle-free, there exists a vertex u of R_1 with fewer than $0.3465d$ out-neighbors in R_1, and then we have $r_2 > 0.6535d$, hence $r_1 + r_2 > 1.6535d$. Now, if $r_3 \geq d$, it follows $r_3' \geq 2.6535d$, and the assertion is proved. Suppose now that $r_3 < d$. It is easy to see that a vertex of R_2 has all its out-neighbors in R_3'. It follows that a vertex of R_2 has at least $d - r_3$ out-neighbors in R_2'. Since every vertex of R_1 has all its out-neighbors in R_2', it follows $a(D[R_2']) \geq r_1 d + r_2(d - r_3)$, hence :

$$a(D[R_2']) \geq r_1 d + r_2 d - r_2 r_3 \qquad (3)$$

On the other hand by Theorem 1.7, we have

$$a(D[R_2']) \leq \frac{(r_2')^2}{2} - \frac{(d - r_3)^2}{1.7232} \qquad (4)$$

From (3) and (4), we deduce $r_1 d + r_2 d - r_2 r_3 \leq \dfrac{r_1^2 + r_2^2 + 2r_1 r_2}{2} - \dfrac{d^2 - 2dr_3 + r_3^2}{1.7232}$, hence $3.4464 r_1 d + 3.4464 r_2 d - 3.4464 r_2 r_3 \leq 1.7232 r_1^2 + 3.4464 r_1 r_2 + 1.7232 r_2^2 - 2d^2 + 4r_3 d - 2r_3^2$. An easy calculation yields : $1.7232(r_2 + r_3 + r_1 - d)^2 \geq 3.7232 r_3^2 - (7.4464d - 3.4464 r_1) r_3 + 3.7232 d^2$. Since $r_1 \geq d$, we get $1.7232(r_2 + r_3 + r_1 - d)^2 \geq 3.7232 r_3^2 - 4r_3 d + 3.7232 d^2$, that is $1.7232(r_2 +$

$r_3 + r_1 - d)^2 \geq f(r_3)$, f being the function defined by $f(t) = 3.7232t^2 - 4dt + 3.7232d^2$. By a classical result on the functions of second degree, we have $f(r_3) \geq f\left(\frac{2d}{3.7232}\right)$, hence $f(r_3) > 2.648d^2$. We deduce then $1.7232(r_2 + r_3 + r_1 - d)^2 > 2.648d^2$, hence $r_2 + r_3 + r_1 - d > 1.239d$ which yields $r'_3 > 2.239d$, and the assertion is still proved. Similarly, we have $r'_{-3} > 2.239$. Since D is triangle-free, by Theorem 1.3, the strong connectivity k of D verifies $k > \dfrac{2-5c}{2-c}d$, and since $c \leq 0.3465$, we get $k > 0.161d$. It is clear that each of the sets R_4, R_5 and R_6 disconnects D, and then $r_i > 0.161d$ for $4 \leq i \leq 6$. Suppose that $r_4 < 0.205d$. Then $D[R'_3]$, which is triangle-free, is of minimum out degree at least $0.795d$. It follows $0.795 < 0.3465r'_3$, hence $r'_3 > 2.2943d$. We have then $v(D) > 2.2943d + 2.239d + 3 \times 0.161d$, that is $v(D) > 5.0163d$, which is not possible. It follows $r_4 \geq 0.205d$. We deduce then $v(D) > 2.239d + 2.239d + 0.205d + 2 \times 0.161d$, that is $v(D) > 5.005d$, which is still impossible. Consequently, the diameter of D is at most 9, and the result is proved. ■

4 An open problem

Theorem 1.3 gives rise to the following question :

Open Problem . For r with $2 < r < \frac{2}{c}$, what is the maximum number $\psi(r) \in]0, 1]$ such that every oriented graph D of minimum semi-degree d of order $n \leq rd$ and of connectivity $k(D) \leq \psi(r)d$, contains a triangle ?

By the result of [8], we have $\psi(r) = 1$ for $2 < r \leq 2.91082$. By Theorem 1.3, for $2.91082 < r < \frac{2}{c}$ we have $\psi(r) \geq \max\left\{\dfrac{5-r-4c+c^2}{(1-c)(2-c)}d, \dfrac{2-cr}{2-c}d\right\}$. Thus, since $c \leq 0.3465$, we get $\psi(3) > 0.679$, $\psi(3.5) > 0.476$, $\psi(4) > 0.371$ $\psi(4.5) > 0.266$, $\psi(5) > 0.161$ and $\psi(5.5) > 0.057$. Observe that Conjecture 1.1 is true, if and only if $\psi(3) = 1$.

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